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# EXTREMAL STRATEGIES IN DIFFERENTIAL GAMES WITH INTEGRAL CONSTRAINTS

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The game problem is considered of taking a controlled motion onto a given set. It is assumed that the players' controls are subject to integral constraints. The first player's extremal strategy is described, forming a feedback control. It is shown that under the conditions of absorption stability the extremal strategy guarantees the termination of the pursuit at the instant of program absorption. A modification is suggested of the extremal strategies, described in [1, 2], in differential games with constraints on the instantaneous values of the players' controls. This paper is closely related to [2 - 6].

**1.** Suppose that the motion of a controlled system is described by the differential equation

$$dx/dt = A(t)x + B(t)u - C(t)v, \quad x[t_0] = x_0 \tag{1.1}$$

Here $x = \{x_1, x_2, \ldots, x_n\}$ is the $n$-dimensional phase vector of the system; $u$, $v$ are controls of dimension $r$; $A(t)$, $B(t)$, $C(t)$ are matrices of corresponding dimensions, depending continuously on $t$. The realizations $u[t]$, $v[t]$ of the controls are subject on $[t_0, \infty)$ to the conditions

$$\int_{t_0}^{\infty} \|u[\xi]\|^2 d\xi \leqslant \mu^2[t_0], \qquad \int_{t_0}^{\infty} \|v[\xi]\|^2 d\xi \leqslant \nu^2[t_0] \tag{1.2}$$

Here $\mu[t_0]$, $\nu[t_0]$ are constraints on the control resources. We assume further that variations of the quantities $\mu[t]$, $\nu[t]$ are determined by the expendable resources

$$\mu^2[t+\Delta] = \mu^2[t] - \int_t^{t+\Delta} \|u[\xi]\|^2 d\xi$$

$$\nu^2[t+\Delta] = \nu^2[t] - \int_t^{t+\Delta} \|v[\xi]\|^2 d\xi$$

With these variations we associate the differential equations

$$d\mu^2/dt = -\|u[t]\|^2, \qquad d\nu^2/dt = -\|v[t]\|^2 \tag{1.3}$$

Thus, system (1.1) with constraints (1.2) has been associated with the system of differential equations (1.1), (1.3) with the initial conditions

$$\mu^2[t_0] = \mu_0^2, \; \nu^2[t_0] = \nu_0^2, \; x[t_0] = x_0.,$$

Motion is considered in $R^{n+2}$ an $(n+2)$-dimensional Euclidean space of points $p = \{\sqrt{\mu^2}, \sqrt{\nu^2}, x_1, x_2, \ldots, x_n\}$. The first player's problem is to choose a control

$u[t]$ so as to lead the point $p[t]$ onto a set $M^*$, where $M^* = \{\mu \geqslant 0, \nu \geqslant 0, M\}$. Here $M$ is a given bounded convex set from $R^n$. By choosing $v[t]$ the second player tries to prevent the point $p[t]$ from going onto set $M^*$. It is reckoned that at the instant $t$ the players know the position $(t, p[t])$ but do not know the control chosen by the opponent at the given and at the next instants.

We consider the problem facing the next player: to construct a control $u_e(t, p)$ such that, for any admissible control $v(t, p)$ it ensures that the point $p[t]$ is led onto $M^*$ at a certain instant $t = \vartheta$. The definition of the class of admissible controls and of the solutions of system (1.1), (1.3) corresponding to them is given below. We present auxiliary statements which will be used in the proof of the fundamental theorem contained in Sect. 4.

Definition 1.1. Every vector-valued function $u(\xi)$ $(v(\xi))$ with values of the vector $u(\xi)$ $(v(\xi))$ from $R^r$, satisfying condition (1.2), is called the first (second) player's program control admissible for the point $p[t_0] = \{\mu[t_0], \nu[t_0], x[t_0]\}$.

By $W(t, \vartheta)$ we denote the set of all points $p = \{\mu, \nu, x\}$ possessing the following property: for any program control $v(\xi)$ $(t \leqslant \xi \leqslant \vartheta)$, admissible for point $p$, we can find a program control $u(\xi)$ $(t \leqslant \xi \leqslant \vartheta)$ of the first player, admissible for point $p$, such that the pair $u(\xi)$, $v(\xi)$ takes the system (1.1), (1.3) from the state $p[t] = p$ into a certain state $p[\vartheta]$ such that $p[\vartheta] \in M^*$. It is not difficult to verify that the inclusion $p \in W(t, \vartheta)$ is equivalent to the following inequality (for example, see [1, 2]):

$$f'(l; t, \vartheta)\, p - \rho_{-M}(l) \leqslant 0 \qquad \text{for } \|l\| = 1$$

Here, $\rho_{-M}(l) = \max_{q \in -M} q'l$ is the support function for the set $-M$, the prime denotes transposition, $l$ is a vector of space $R^n$,

$$f(l;\ t,\ \vartheta) = \{-\rho_1(l;\ t,\ \vartheta),\ \rho_2(l;\ t, \vartheta),\ -l'X[\vartheta, t]\}$$

is a vector of space $R^{n+2}$. We assume here that the following conditions are fulfilled: for any two instants $t_1$, $t_2$ $(t_1 < t_2)$ the inequalities

$$\rho_i(l; t_1, t_2) > 0$$

$$\rho_i(l; t, \vartheta) = \Big(\int_t^{\vartheta} \|l'H_i[\vartheta, \xi]\|^2 d\xi\Big)^{1/2} \qquad (i = 1, 2)$$

$$H_1[\vartheta, \xi] = X[\vartheta, \xi]B(\xi), \quad H_2[\vartheta, \xi] = X[\vartheta, \xi]\, C(\xi)$$

where $X[\vartheta; \xi]$ is the fundamental matrix of system (1.1), are valid for any vector $l$ $(l \in R^n,\ l \neq 0)$.

Definition 1.2. The smallest value of parameter $\vartheta$ for which the inclusion $p \in W(t, \vartheta)$ holds is called the absorption instant $\vartheta^\circ = \vartheta^\circ(t, p)$ corresponding to the position $(t, p,)$.

This definition is well-posed in the sense that the existence of the smallest instant $\vartheta^\circ$ satisfying the inclusion follows from the existence of instants $\vartheta$ satisfying the inclusion $p \in W(t, \vartheta)$ and from the continuous dependence of the vector-valued function $f(l; t, \vartheta)$ on the collection $l, t, \vartheta$.

We accept the following conditions as fulfilled.

Condition 1.1. There exists an absorption instant $\vartheta^\circ = \vartheta(t_0, p_0) \geqslant t_0$

corresponding to the game's initial position $(t_0, p_0)$.

Condition 1.2. The absorption is strongly $u$-stable, i.e., for any point $p$ $(p \in W(t, \vartheta^\circ))$ and any program $v(\xi)$ $(t \leqslant \xi \leqslant t + \Delta)$ admissible for $p$, we can find a program control $u(\xi)$ $(t \leqslant \xi \leqslant t + \Delta)$ such that the pair $(u(\xi), v(\xi))$ transfers the motion corresponding to system (1.1), (1.3) from the position $p = p[t]$ to a certain state $p[t + \Delta]$ such that $p[t + \Delta] \in W(t + \Delta, \vartheta^\circ)$. This condition must be fulfilled for all $t$, $\Delta$, where $t_0 \leqslant t \leqslant \vartheta^\circ$ $(0 \leqslant \Delta \leqslant \vartheta^\circ - t)$.

**2.** We state certain properties of the sets $W(t, \vartheta^\circ)$, following from Conditions 1.1, 1.2 and from Definition 1.2.

Property 2.1. The set $W(t, \vartheta^\circ)$ is nonempty, convex and closed for any $t$ $(t_0 \leqslant t \leqslant \vartheta^\circ)$.

Property 2.2. The equality $W(\vartheta^\circ, \vartheta^\circ) = M^*$ is valid. The set $W(t, \vartheta^\circ)$ is unbounded. For what is to follow we accept to consider those and only those points of set $W(t, \vartheta^\circ)$(and of other auxiliary sets) whose first coordinate is less than or equal to some fixed positive number $\mu_0$. Such an assumption is based on the fact that the first coordinate of the motion being considered does not increase with time.

Property 2.3. For any $t \in [t_0, \vartheta^\circ)$ the set $W(t, \vartheta^\circ)$ depends continuously on $t$ with respect to inclusion and is upper-semicontinuous with respect to inclusion from the left in the variable $t$ at the point $t = \vartheta^\circ$, i.e., if $t_k \to \vartheta^\circ$, $t_k < \vartheta^\circ$, $p_k \in W(t_k, \vartheta^\circ)$ and $\lim\limits_{k\to\infty} p_k = p_*$, then $p_* \in W(\vartheta^\circ, \vartheta^\circ)$.

Proof. We first consider the case $t < \vartheta^\circ$. As a preliminary we prove the following assertion.

Lemma 2.3.1. For any $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0$ such that the inclusion $W(t - \Delta, \vartheta^\circ) \subset W_\varepsilon(t, \vartheta^\circ)$, where $W_\varepsilon(t, \vartheta^\circ)$ is the $\varepsilon$-neighborhood of set $W(t, \vartheta^\circ)$ is true for all $\Delta$ $(0 \leqslant |\Delta| \leqslant \delta(\varepsilon))$.

To prove Lemma 2.3.1 we consider the set $W^\omega(t, \vartheta^\circ)$ of points $p$ defined by the inequality

$$f'(l; t, \vartheta)p - \rho_M(l) \leqslant \omega \qquad (0 \leqslant \omega \leqslant \infty) \quad \text{for } \|l\| = 1$$

The assertion

$$\lim_{\omega\to 0} W^\omega(t, \vartheta^\circ) = W(t, \vartheta^\circ) \tag{2.1}$$

is valid. Equality (2.1) follows from the definition of $W^\omega(t, \vartheta^\circ)$ and from the constraint imposed on the first coordinates of the points of sets. From the assumption made on the boundedness of the coordinates of the points $p$ follows the equiboundedness of the sets $W(t - \Delta, \vartheta^\circ)$, where $|\Delta| \leqslant \Delta_0$, $0 < \Delta_0 < \vartheta^\circ - t$. Taking this into account and the continuous dependence of the function $f(l; t, \vartheta)$ on the variables $l$, $t$, we get that for any $\omega > 0$ we can find $\delta(\omega) > 0$ $(\delta(\omega) \leqslant \Delta_0)$ such that the inequality

$$|(f(l; t, \vartheta^\circ) - f(l; t - \Delta, \vartheta^\circ))' q| \leqslant \omega \quad \text{for } \|l\| = 1$$

is true for all $\Delta$ $(|\Delta| \leqslant \delta(\omega))$ and for all $q$ $(q \in W(t - \Delta, \vartheta^\circ))$ and, consequently, the inclusion $W(t - \Delta, \vartheta^\circ) \subset W^\omega(t, \vartheta^\circ)$ is true. The validity of Lemma 2.3.1 follows from this and from condition (2.1).

The following assertions are valid: for any $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0$ such that the inclusions

$$W(t, \vartheta^\circ) \subset W_\varepsilon(t + \Delta, \vartheta^\circ), \qquad W(t, \vartheta^\circ) \subset W_\varepsilon(t - \Delta, \vartheta^\circ)$$

are true for all $\Delta$ $(0 \leqslant \Delta \leqslant \delta(\varepsilon))$. The proof of these assertions is based on Conditions

1.1, 1.2, and on the fact that the set $W(t, \vartheta^\circ)$ contains an interior point.

We now consider the case $t = \vartheta^\circ$. Let us prove the semicontinuity of the set $W(t, \vartheta^\circ)$ for $t = \vartheta^\circ$ as $t$ varies from the left. We take an arbitrary convergent sequence of points $p_k$ $(p_k \in W(t_k, \vartheta^\circ))$, where $t_k < \vartheta^\circ$ and $t_k \to \vartheta^\circ$ as $k \to \infty$. The inequality

$$-\mu_k \rho_1(l; t_k, \vartheta^\circ) + \nu_k \rho_2(l; t_k, \vartheta^\circ) - l' X[\vartheta^\circ, t_k] x_k \leqslant \rho_{-M}(l)$$

is valid for $\|l\| = 1$. Passing to the limit as $k \to \infty$, we obtain

$$-l' \lim_{k\to\infty} x_k \leqslant \rho_{-M}(l) \quad \text{for } \|l\| = 1$$

Hence it follows that

$$\lim_{k\to\infty} p_k \in M^* = W(\vartheta^\circ, \vartheta^\circ)$$

We have proven the validity of Property 2.3.

By $\varepsilon[t, p]$ we denote the distance from point $p$ to the convex set $W(t, \vartheta^\circ)$

$$\varepsilon[t, p] = \begin{cases} \varkappa[t, p] & \text{for } \varkappa[t, p] > 0 \\ 0 & \text{for } \varkappa[t, p] \leqslant 0 \end{cases}$$

$$\varkappa[t, p] = \max_s \{s'p - \rho(s; t, \vartheta^\circ)\} \quad \text{for } \|s\| = 1 \tag{2.2}$$

$$\rho(s; t, \vartheta^\circ) = \max_w s'w \quad \text{for } w \in W(t, \vartheta^\circ),\ s = \{s_1, s_2, s^*\}\ (s^* \in R^n)$$

Property 2.4. The function $\varepsilon = \varepsilon[t, p]$ is continuous in the collection $\{t, p\}$ for all $p$, $t_0 \leqslant t \leqslant \vartheta^\circ$

Property 2.5. If $\varepsilon[t, p] > 0$, then the maximum in (2.2) is achieved on a single vector $s = s(t, p)$.

Property 2.6. Let $\varepsilon[t_*, p_*] > 0$. Then the vector-valued function $s = s(t, p)$ is continuous in $\{t, p\}$ in the neighborhood of the point $(t_*, p_*)$.

The validity of Properties 2.4, 2.6 follows from the continuity of the variation of set $W(t, \vartheta^\circ)$ as $t$ varies. The validity of Property 2.5 follows from the condition of convexity of set $W(t, \vartheta^\circ)$ in the space $R^{n+2}$.

3. Let us make the preliminary problem statement more precise.

Definition 3.1. Let $\{U\}$ be a collection of closed convex sets $U$ of an $r$-dimensional vector space. The function $U = U(t, p)$, which associates a certain set $U$ from $\{U\}$ with every vector $\{t, p\}$ is called an admissible control of the first player if:

1) $U(t, p)$ depends upper-semicontinuously relative to inclusion on the collection $(t, p)$;

2) for any $t_1 < \vartheta^\circ$ and any bounded closed region $D$ of the set $[t_0, t_1] \times (p_1 \geqslant 0, p_2 \geqslant 0, p_3, \ldots, p_{n+2})$ there exists a summable function $B_*(t)$ such that the condition: if $u \in U(t,p)$, then $\|u\|^2 \leqslant B_*(t)$, is fulfilled almost everywhere in $D$

3) $U(t, p) = 0$ for $p_1 < 0$. Here $U(t, p)$ was fully determined in formal fashion in the region where $p_1 < 0$.

The second player's admissible control $V(t, p)$ is defined analogously.

Definition 3.2. Every absolutely continuous vector-valued function $\{p_1^2[t], p_2^2[t], q[t]\}$ taking at $t = t_0$ a specified value $\{p_1^2[t_0], p_2^2[t_0], q[t_0]\} = \{p^2_{10}, p^2_{20}, q_0\}$ and satisfying for almost all $t$, $t_0 \leqslant t \leqslant t_1$, the condition

$$dq[t]/dt = A(t) q[t] + B(t) u[t] - C(t) v[t] \tag{3.1}$$
$$dp_1^2[t]/dt = -\|u[t]\|^2, \quad dp_2^2[t]/dt = -\|v[t]\|^2$$

where the summable functions $u[t]$, $v[t]$ satisfy the inclusions

$$u[t] \in U(t, p[t]), \qquad v[t] \in V(t, p[t])$$
$$p[t] = \{\mu[t], \nu[t], x[t]\} = \{p_1[t], p_2[t], q[t]\}$$

is called a solution of system (1.1), (1.3), generated by the pair of admissible controls $U(t, p)$, $V(t, p)$ on the interval $[t_0, t_1]$ $(t_1 < \vartheta^\circ)$ .

Since the solution of system (1.1), (1.3), generated by a pair of admissible strategies $U(t, p)$, $V(t, p)$, has been defined on any interval $[t_0, t_1]$ $(t_1 < \vartheta^\circ)$, it can be fully determined with respect to continuity at the point $\vartheta^\circ$. The existence of a solution and its continuability to the interval $[t_0, \vartheta^\circ]$ follow from the results in [7] (see Theorems 3, 4). Thus, we have defined a solution of system (1.1), (1.3) on the interval $[t_0, \vartheta^\circ]$ generated by a pair of admissible strategies $U(t, p)$, $V(t, p)$.

The problem. Construct an admissible control $U = U_e(t, p)$ such that for any solution of system (3.1), generated by the pair $U = U_e(t, p)$ and $V = V(t, p)$ (here $V(t, p)$ is an arbitrary admissible control), the condition $p[t] \in M^*$ must be realized no later than at some finite instant $t = \vartheta^\circ$.

**4.** We define $U_e(t, p)$ for instants $t < \vartheta^\circ$ in the following way:

$$U_e(t,p) = \frac{B'(t)\, s^*(t, p)}{s_1(t, p)}\, p_1, \quad \text{if} \quad \varkappa[t,p] > 0, \ p_1 > 0$$

$$U_e(t, p) = 0, \quad \text{if} \quad \varkappa[t, p] \geqslant 0 \text{ and } p_1 \leqslant 0 \quad \text{or} \quad \varkappa[t, p] < 0$$

$$U_e(t, p) = \mathrm{CO} \bigcup_{s(t,p) \in \Omega} \frac{B'(t)\, s^*(t, p)}{s_1(t, p)}\, p_1, \quad \text{if} \quad \varkappa[t, p] = 0, p_1 > 0$$

Here, CO denotes the closed convex hull, $\Omega$ is the union of the set of vectors $s(t, p)$ satisfying condition (2.2) with the 0- vector of space $R^r$. The following assertion is valid: the vector $s(t, p)$ satisfying condition (2.2) is such that $s_1(t,p) < 0$ in case $t < \vartheta^\circ$.

From Property 2.6 and the inequality $s_1(t, p) < 0$ $(t < \vartheta^\circ)$ it follows that the set $U_e(t, p)$ is upper-semicontinuous relative to inclusion for a variation of position $(t,p)$ and also the set $U_e(t,p)$ is equibounded on any compact set $D$ from the set $[t_0, t_1] \times \times R^{n+2}$ where $t_1$ is an arbitrary number satisfying the inequality $t_1 < \vartheta^\circ$. Furthermore, $U_e(t, p) = 0$ for $p_1 < 0$. Hence it follows that $U_e(t, p)$ is an admissible control in the class of feedback controls $U(t, p)$ .

Theorem 4.1. Suppose that Conditions 1.1, 1.2 are satisfied for the position $(t_0, p_0)$ then the control $U_e(t, p)$ guarantees that the system (1.1), (1.3) is led from the initial state $p_0 = p[t_0]$ onto the set $M^*$ at no later than the instant $\vartheta^\circ$.

Proof. Let us investigate the variation of the quantity $\varepsilon[t, p[t]_{U_e, V}]$ on the interval $[t_0, \vartheta^\circ]$ ; here $p[t]$ is the motion generated by the control $U_e(t, p)$ in conjunction with an arbitrary admissible $V(t, p)$. We show that $\varepsilon[t, p[t]_{U_e, V}] \equiv 0$ on $[t_0, \vartheta]$ for any $\vartheta < < \vartheta^\circ$. We assume the contrary: there exists the second player's admissible strategy $V(t, p)$ such that the pair $U_e(t, p)$, $V(t, p)$ generates a solution $p = p[t]$ on $[t_0, \vartheta]$, where $\vartheta < \vartheta^\circ$ and $u_e[t] = u_e(t, p[t]) \in U_e(t, p[t])$, $v[t] \in V(t, p[t])$ are such that $\varepsilon[t, p[t]_{u_e, v}] \not\equiv 0$ on $[t_0, \vartheta]$. Hence it follows that there exist $\tau \in [t_0, \vartheta]$ and $\Delta_*$ $(\tau + \Delta_* < \vartheta)$ such that the conditions

$$p[\tau]_{w_e, \vartheta} \in W(\tau, \vartheta^\circ), \quad p[\tau + \Delta]_{u_e, v} \notin W(\tau + \Delta, \vartheta^\circ)$$

are fulfilled for any $\Delta$ $(0 < \Delta \leqslant \Delta_*)$. Here the symbol $W^{\cdot}(\tau, \vartheta^{\circ})$ denotes the boundary of the set $W(\tau, \vartheta^{\circ})$ in the space $R^{n+2}$. The following assertion is valid.

Lemma 4.1.1. We can find $\Delta_0$ $(0 < \Delta_0 < \Delta_*)$ and $K$ $(0 < K < \infty)$ such that the inequality

$$I_1(\Delta) \leqslant K(I_2(\Delta) + \Delta), \quad I_1(\Delta) = \int_t^{t+\Delta} \|u_\Delta(\xi)\|^2 d\xi, \quad I_2(\Delta) = \int_t^{t+\Delta} \|v[\xi]\|^2 d\xi$$

is valid for any $t \in [\tau, \tau + \Delta_0]$ and the corresponding point $p_*[t]_{u_e, v} \in W^{\cdot}(t, \vartheta^{\circ})$, for any $\Delta$ $(0 < \Delta \leqslant \Delta_0)$, and for any $u_\Delta(\xi)$ $(t \leqslant \xi \leqslant t + \Delta)$.

Here, by $p_*[t]_{u_e, v}$ we have denoted a point of set $W^{\cdot}(t, \vartheta^{\circ})$ nearest to the point $p[t]_{u_e, v}$. Recall also that the symbol $u_\Delta(\xi)$ $(t \leqslant \xi \leqslant t + \Delta)$ denotes a control which in pair with $v[\xi]$ $(t \leqslant \xi \leqslant t + \Delta)$ satisfies Condition 1.2.

The proof of Lemma 4.1.1 relies on the construction of sets $W(t + \Delta, \vartheta^{\circ})$ $(0 \leqslant \Delta \leqslant \Delta_*)$ and on Conditions 1.1, 1.2, and is carried out by contradiction. We make use of Lemma 4.1.1 to estimate the quantity $\varepsilon^2[t + \Delta, p[t + \Delta]_{u_e, v}]$ at the points $t \in [\tau, \tau + \Delta_0]$ $(0 < \Delta \leqslant \Delta_0)$.

We first make some auxiliary estimates. The equality

$$\begin{aligned} \|p^*{}_2[t + \Delta]_{u_\Delta, v} - p[t + \Delta]_{u_\Delta, v}\|^2 &= |p_1^*[t + \Delta]_{u_\Delta} - p_1[t + \Delta]_{u_\Delta}|^2 + \\ &+ |p_2^*[t + \Delta]_v - p_2[t + \Delta]_v|^2 + \|q^*[t + \Delta]_{u_\Delta, v} - q[t + \Delta]_{u_\Delta, v}\|^2 \end{aligned} \tag{4.2}$$

is valid. Here $p[t + \Delta]_{u_\Delta, v}$ $(p^*[t + \Delta]_{u_\Delta, v})$ is the state at instant $t + \Delta$ of the motion of (1.1), (1.3) with the initial condition $p[t]_{u_e, v}$ $(p_*[t]_{u_e, v})$, generated by the pair $u_\Delta(\xi)$, $v[\xi]$ $(t \leqslant \xi \leqslant t + \Delta)$. The following assertions are valid: there exist $\lambda_1, \lambda_2, \lambda_3$ $(0 < \lambda_1, \lambda_2, \lambda_3 < \infty)$ such that the inequalities

$$\begin{aligned} |p_2^*[t + \Delta]_v - p_2[t + \Delta]_v|^2 &= |(p_{2*}[t] - I_2(\Delta))^{1/2} - (p_2[t]^2 - I_2(\Delta))^{1/2}| \leqslant \\ &\leqslant |p_{2*}[t] - p_2[t]|^2 \exp \lambda_1 I_2(\Delta) \end{aligned} \tag{4.3}$$

$$\|q^*[t + \Delta]_{u_\Delta, v} - p[t + \Delta]_{u_\Delta, v}\|^2 \leqslant \|q_*[t] - q[t]\|^2 \exp \lambda_2 \Delta \tag{4.4}$$

$$|p_1^*[t + \Delta]_{u_\Delta} - p_1[t + \Delta]_{u_\Delta}|^2 \leqslant |p_{1*}[t] - p_1[t]|^2 \exp \lambda_3 (I_2(\Delta) + \Delta) \tag{4.5}$$

are fulfilled for any $t \in [\tau, \tau + \Delta_0]$ and $\Delta$ $(0 < \Delta \leqslant \Delta_0)$. The validity of the next lemma follows from relations (4.2) - (4.5).

Lemma 4.1.2. There exists $\lambda$ $(0 < \lambda < \infty)$ such that the inequality

$$\|p^*[t + \Delta]_{u_\Delta, v} - p[t + \Delta]_{u_\Delta, v}\|^2 \leqslant \|p_*[t] - p[t]\|^2 \exp \lambda (I_2(\Delta) + \Delta)$$

is valid for any $t \in [\tau, \tau + \Delta_0]$, for any $\Delta$ $(0 < \Delta \leqslant \Delta_0)$, and for any $u_\Delta(\xi)$ $(t \leqslant \xi \leqslant t + \Delta)$.

We separate the interval $[\tau, \tau + \Delta_0]$ into two nonintersecting sets $M_1$ and $M_2$. Here $M_1$ is the set of points $t$ such that there exists a number $L = L(t)$ $(0 < L < \infty)$ and a sequence $\{\Delta_n;\ \Delta_n > 0,\ \Delta_n \to 0 \text{ as } n \to \infty\}$ such that the inequality

$$\frac{1}{\Delta_n} I_2(\Delta_n) \leqslant L \tag{4.6}$$

is true, and $M_2$ is the set of points $t$ such that the equality

$$\lim_{\Delta \to 0} \frac{1}{\Delta} I_2(\Delta) = \infty \tag{4.7}$$

is valid. The relation

$$\varepsilon[t+\Delta,\ p[t+\Delta]_{u_e,v}]-\varepsilon[t+\Delta,\ p[t+\Delta]_{u_\Delta,v}]\leqslant \leqslant s'(t+\Delta,\ p[t+\Delta]_{u_e,v})(p[t+\Delta]_{u_e,v}-p[t+\Delta]_{u_\Delta,v}) \tag{4.8}$$

is valid in case $t\in M_1$. Here $s(t+\Delta,\ p[t+\Delta]_{u_e,v})=\{s_1(t+\Delta,\ p[t+\Delta]_{u_e,v}),$ $s_2(t+\Delta,\ p[t+\Delta]_{u_e,v}),\ s^*(t+\Delta,\ p[t+\Delta]_{u_e,v})\}$ is a unit vector of space $R^{n+2}$, satisfying the condition (see Sect. 2)

$$s'(t+\Delta,\ p[t+\Delta]_{u_e,v})\,p[t+\Delta]_{u_e,v}-\rho(s(t+\Delta,\ p[t+\Delta]_{u_e,v});\ t+\Delta,\ \vartheta^0)= =\varkappa[t+\Delta,\ p[t+\Delta]_{u_e,v}]$$

From the continuity of the vector-valued functions $s^*(t+\Delta,p[t+\Delta]_{u_e,v})$ and $s_1(t+\Delta,p[t+\Delta]_{u_e,v})$ in $\Delta$ at the point $\Delta=0$, the continuity of the function $u_e(\xi,p[\xi])$ with respect to position at the point $(t,p[t]_{u_e,v})$, and the continuity of matrix $B(\xi)$ : it follows that the right-hand side of inequality (4.8) equals the expression

$$\int_t^{t+\Delta} s^{*\prime}(t,\ p[t]_{u_e,v})\,B(t)\,(u_e[t]-u_\Delta(\xi))\,d\xi+s_1(t,\ p[t]_{u_e,v})\times \times\left\{-\frac{\|u_e[t]\|^2}{2p_1[t]_{u_e}}\Delta+o_1(\Delta)+\frac{1}{2p_1[t]_{u_e}}I_1(\Delta)+o(I_1(\Delta))\right\}+o_2(\Delta) \tag{4.9}$$

Here and subsequently, $o$, $o_1$, $o_2$ are quantities infinitesimal in comparison with the quantities standing alongside in the braces.

From inequality (4.5) and Lemma 4.1.1 it follows that when $\Delta=\Delta_n$ expression (4.9) equals

$$\int_t^{t+\Delta_n}\Big\{s^{*\prime}(t,\ p[t]_{u_e,v})\,B(t)\,(u_e[t]-u_{\Delta_n}(\xi))+ +s_1(t,\ p[t]_{u_e,v})\left(-\frac{\|u_e[t]\|^2}{2p_1[t]_{u_e}}+\frac{\|u_{\Delta_n}(\xi)\|^2}{2p_1[t]_{u_e}}\right)\Big\}\,d\xi+o(\Delta_n) \tag{4.10}$$

From the definition of the control $U_e(t,p)$ it follows that the integrand in (4.10) is non-positive. Then, by virtue of (4.8), (4.9), we obtain the estimate

$$\varepsilon[t+\Delta_n,\ p[t+\Delta_n]_{u_e,v}]-\varepsilon[t+\Delta_n,\ p[t+\Delta_n]_{u_{\Delta_n}v}]\leqslant o(\Delta_n)$$

Here $o(\Delta_n)$ is an infinitesimal depending on $t$. From this inequality and from Lemma 4.1.2 it follows that in case $t\in M_1$ we can find $\lambda$ $(0<\lambda<\infty)$ and a sequence $\{\Delta_n\}$ $(\Delta_n>0,\ \Delta_n\to 0$ as $n\to\infty)$ such that the inequality

$$\varepsilon^2[t+\Delta_n,\ p[t+\Delta_n]_{u_e,v}]\leqslant\varepsilon^2[t,\ p[t]_{u_e,v}]\exp\lambda(I_2(\Delta_n)+\Delta_n)$$

is true.

In case $t\in M_2$ the inequality

$$\varepsilon^2[t+\Delta,\ p[t+\Delta]_{u_e,v}]-\varepsilon^2[t,\ p[t]_{u_e,v}]\leqslant 0$$

is true for all sufficiently small $\Delta>0$. The proof of this assertion relies essentially on the limit relation (4.7) as well as on the boundedness of the control $u_e[\xi]$ $(t\leqslant\xi\leqslant\leqslant t+\Delta_n)$. The validity of the next lemma follows from everything we have said.

Lemma 4.1.3. The inequality

$$\varepsilon^2[\tau+\Delta_0,\ p[\tau+\Delta_0]_{u_e,v}] \leqslant \varepsilon^2[t_*,\ p[t_*]_{u_e,v}]\exp(\lambda I)$$

$$I = \int_{t_*}^{\tau+\Delta_0} \|v[\xi]\|^2 d\xi + (\tau+\Delta_0 - t_*)$$

is true for any point $t_* \in (\tau, \tau+\Delta_0]$.

The validity of the equality $\varepsilon[\tau+\Delta_0,\ p[\tau+\Delta_0]_{u_e,v}] = 0$ follows from Lemma 4.1.3, which contradicts the assumption made earlier.

The case $p_{1*}[\tau] \neq 0,\ p_{2*}[\tau] \neq 0$ was considered above. Let us consider the rest of the possible cases. When $p_{1*}[\tau] = 0$, the validity of Theorem 4.1 is proved by contradiction. In case $p_{2*}[\tau] = 0$ we can prove the validity of the assertion: we can find $K$ $(0 < K < \infty)$ and $\Delta_0$ $(0 < \Delta_0 < \Delta_*)$ such that the inequality $I_1(\Delta) \leqslant K\Delta$ is true for any $t \in (\tau, \tau+\Delta_0]$ and the corresponding point $p_*[t]_{u_e,v} \in W^*(t, \vartheta^\circ)$ and for any $u_\Delta(\xi)$ $(t \leqslant \xi \leqslant t+\Delta,\ 0 < \Delta \leqslant \Delta_0)$.

From this assertion follows the inequality

$$\varepsilon^2[\tau+\Delta_0,\ p[\tau+\Delta_0]_{u_e,v}] \leqslant \varepsilon^2[t_*, p[t_*]_{u_e,v}]\exp\lambda\Delta_0$$

for any $t_* \in (\tau, \tau+\Delta_0]$, where $\lambda$ is some number $(0 < \lambda < \infty)$. Hence follows the equality $\varepsilon[\tau+\Delta_0, p[\tau+\Delta_0]_{u_e,v}] = 0$, contradicting condition 4.1. Theorem 4.1 is proved.

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